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SPACES WITH LARGE PROJECTION CONSTANTS

BY
HERMANN KÖNIG
Mathematisches Seminar, Universität Kiel, 23 Kiel, W. Germany

ABSTRACT

For every prime number k, we give an explicit construction of a complex k-dimensional space X_k with projection constant $\gamma(X_k)=\sqrt{k}-1/\sqrt{k}+1/k$. Moreover, there are real k-dimensional spaces X_k with $\gamma(X_k)\geqq\sqrt{k}-1$ for a subsequence of integers k. Hence in both cases $\gamma(X_k)/\sqrt{k}\to 1$ which is the maximal possible value since $\gamma(X_k)\leqq\sqrt{k}$ is generally true.

If X is a closed subspace of a Banach space Y, the relative projection constant of X in Y is defined by

$$\gamma(X, Y):=\inf\{\|P\| \mid P: Y\to Y \text{ is a projection onto } X\},$$

and the projection constant of a Banach space X by

$$\gamma(X):=\sup\{\gamma(X, Y) \mid Y \text{ is a Banach space containing } X \text{ as a subspace}\}.$$

In the case of finite-dimensional spaces, we indicate the dimensions of the spaces by subscripts. Thus $X_k\subseteq Y_n$ means a k-dimensional subspace of an n-dimensional space, $k\leqq n$. It is well-known that $\gamma(X_k)\leqq\sqrt{k}$. In the case that both spaces $X_k\subseteq Y_n$ are finite-dimensional, this was strengthened to

$$\gamma(X_k, Y_n)\leqq f(k,n):=\sqrt{k}\left(\frac{1}{n}\{\sqrt{k}+\sqrt{(n-1)(n-k)}\}\right)$$

(1)

$$\leqq\sqrt{k}\left(1-\frac{(\sqrt{k}-1)^2}{2n}\right),\qquad 1\leqq k\leqq n$$

in [1]. It was an open question whether there is $0<c<1$ such that $\gamma(X_k)\leqq c\sqrt{k}$ holds for any k-dimensional space X_k. By constructing spaces with projection constants as mentioned above, we answer this question negatively. To find

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spaces with very large projection constants, the question of equality $\gamma(X_k, Y_n) = f(k,n)$ in (1) was investigated in Theorem 2 of [1] in the case of $Y_n = l_n^\infty$. The following characterization is an extension of this result.

THEOREM 1. *Let $1 < k < n$. Then the following are equivalent:*

(a) *There is a subspace X_k of l_n^∞ such that $\gamma(X_k) = f(k,n)$.*

(b) *There is an operator $T : l_n^\infty \to l_n^\infty$ with nuclear norm $\nu(T) = 1$ and eigenvalues $\gamma_1(T) = \cdots = \gamma_k(T) = f(k,n)/k$ and $\gamma_{k+1}(T) = \cdots = \gamma_n(T) = (1 - f(k,n))/(n-k)$ — one eigenvalue being k-fold, the other $(n-k)$-fold.*

(c) *There is a projection $P : l_n^\infty \to l_n^\infty$ with $P = (p_{ij})$, $p_{ii} = k/n$ and $|p_{ij}| = 1/n\sqrt{k(n-k)/(n-1)}$ for $i,j = 1,\dots,n$, $i \neq j$.*

(d) *There are "equiangular" vectors $x_1,\dots,x_n \in l_k^2$ with $\|x_i\|_2 = 1$ and $|(x_i, x_j)| = \sqrt{(n-k)/k(n-1)}$ for $i,j = 1,\dots,n$, $i \neq j$.*

(e) *The approximation numbers of $\mathrm{Id}_n : l_n^1 \to l_n^\infty$ are given by*

$$a_{k+1}(\mathrm{Id}_n) = (1 + \sqrt{k(n-1)/(n-k)})^{-1}.$$

Here $(\ ,\ )$ denotes the usual scalar product in $\mathbf{R}^n$ or $\mathbf{C}^n$; theorem 1 holds for real as well as complex spaces. Clearly, matrices P or vectors x_i with (c) or (d) might exist in the complex case without existing in the real case, e.g. for $k = 2$, $n = 4$. Recall the approximation numbers of an operator $T \in L(X, Y)$ are defined by

$$a_k(T) := \inf\{\|T - T_k\| \mid T_k \in L(X,Y),\ \mathrm{rank}\ T_k < k\}, \qquad k \in \mathbf{N}.$$

The best approximation operators in (e) are multiples of the projection P in (c) which projects onto $X_k = P(l_n^\infty)$ with $\|P\| = \gamma(X_k) = f(k,n)$.

PROOF. It was shown in [1] that (a) to (c) are equivalent except that (c) was formulated for reflections $A, A^2 = \mathrm{Id}$, which are related to projections P by $A = 2P - \mathrm{Id}$. Moreover it was shown that A and thus P necessarily have to be selfadjoint.

(c) $\Rightarrow$ (d). Since $\mathrm{tr}(P) = k$ and P (as a projection) only has eigenvalues 0 and 1, $X_k := \mathrm{Im}(P)$ is k-dimensional. Let e_i denote the unit vectors in l_n^∞ and $x_i := \sqrt{n/k}Pe_i$, $i = 1,\dots,n$. Hence $x_i \in X_k$ with

$$|(x_i, x_j)| = n/k\,|(Pe_i, Pe_j)| = n/k\,|(Pe_i, e_j)| = n/k\,|p_{ij}|$$

which is 1 for $i = j$ and equal to $\sqrt{(n-k)/k(n-1)}$ for $i \neq j$.

(d) $\Rightarrow$ (e). Consider

$$U_k := (1 + \sqrt{(n-k)/k(n-1)})^{-1}((x_i, x_j))_{i,j} : l_n^1 \to l_n^\infty.$$

Since $x_i \in l_k^2$, rank $U_k \leqq k$. Thus

$$a_{k+1}(\mathrm{Id}: l_n^1 \to l_n^\infty) \leqq \|\mathrm{Id} - U_k : l_n^1 \to l_n^\infty\|$$

$$= \max_{1 \leqq i,j \leqq n} |\delta_{ij} - (U_k)_{ij}|$$

$$= (1 + \sqrt{k(n-1)/(n-k)})^{-1}.$$

The reverse inequality is always true, cf. [4].

(e) $\Rightarrow$ (c). This was shown in Theorem 3 of Melkman [4]. The best approximation operator also yields a best approximation of $\mathrm{Id}: l_n^1 \to l_n^p$, $2 \leqq p < \infty$; by multiplying with a suitable constant c_p, the best approximation operator is a multiple of the projection P of (c). □

REMARKS. (i) Some cases of (k, n) where matrices P with (c) exist were considered in [1], however no case with $k \to \infty$ and $k/n \to 0$.

(ii) Gerzon observed that n equiangular vectors $x_i \in l_k^2$ can exist only for $n \leqq k(k+1)/2$ in the real and $n \leqq k^2$ in the complex case, cf. Lemmens–Seidel [2]. To see this, note that the selfadjoint maps $P_i := x_i \otimes \bar{x}_i$ are linearly independent in $L(l_k^2)$ since

$$\det(\langle P_i, P_j \rangle) = \det(\mathrm{tr}(P_i P_j)) = \det\begin{pmatrix} 1 & & \alpha^2 \\ & \ddots & \\ \alpha^2 & & 1 \end{pmatrix}$$

$$= (1-\alpha^2)^{n-1}(1+(n-1)\alpha^2) \neq 0$$

with $\alpha^2 := |(x_i, x_j)|^2 < 1$. In the real case, the bound $n = k(k+1)/2$ is attained e.g. for $k = 3$ by the $n = 6$ diagonals of the ikosahedron and for $k = 7$ by the $n = 28$ vectors in the hyperplane $[\Sigma_{i=1}^8 z_i = 0]$ of $\mathbf{R}^8$ having 2 coordinates equal to 3 and 6 coordinates equal to -1.

(iii) In the real case, for $k < n/2$, one has the additional restriction that $\sqrt{k(n-1)/(n-k)}$ should be an odd integer if (a)–(e) hold, cf. [2]. In this case, $f(k, n) \in \mathbf{Q}$. For $k = n/2$ this restriction is not necessary, as the case $k = 3$, $n = 6$ shows $\|x_i\|_2 = 1$, $|(x_i, x_j)| = 1/\sqrt{5}$.

We now construct badly complemented subspaces of l_n^∞ by finding vectors which almost satisfy the equiangularity condition (d).

THEOREM 2. *Let k be a prime and $n = k^2$. Then the vectors*

$$y_j := k^{-1/2}\left(\exp\left(\frac{2\pi i}{k}\{s_1 j + s_2 j^2\}\right)\right)_{s_1, s_2 = 1}^k \in \mathbf{C}^n,$$

$j=1,\dots,k$ span a complex k-dimensional subspace X_k of l_n^∞ with projection constant $\gamma(X_k)=\sqrt{k}-1/\sqrt{k}+1/k$.

LEMMA 1. *Let k be a prime number and $s_1, s_2 \in \mathbf{Z}$ be not both multiples of k. Then*

$$\left|\sum_{j=1}^{k} \exp\left(\frac{2\pi i}{k}(s_1 j + s_2 j^2)\right)\right| \leqq \sqrt{k},$$

for $s_2 = k$, the value of the sum is zero, of course.

This simple lemma can be checked by direct evaluation of the absolute value; it is a special case of a result of A. Weil [7]. An extended version was used by Maiorov [3] to find good estimates for $a_{k+1}(\mathrm{Id}: l_n^1 \to l_n^\infty)$.

PROOF OF THEOREM 2. For $s=(s_1,s_2)\in\{1,\dots,k\}^2$, let

$$x_s := k^{-1/2}\left(\exp\left(\frac{2\pi i}{k}\{s_1 j + s_2 j^2\}\right)\right)_{j=1}^{k} \in \mathbf{C}^k.$$

Then $\|x_s\|_2 = 1$ and for $s=(s_1,s_2)\neq(t_1,t_2)=t$

$$|(x_s,x_t)| = \frac{1}{k}\left|\sum_{j=1}^{k}\exp\left(\frac{2\pi i}{k}\{(s_1-t_1)j+(s_2-t_2)j^2\}\right)\right| \leqq 1/\sqrt{k} \tag{2}$$

by Lemma 1. Consider $P := 1/k((x_s,x_t))_{s,t} : l_n^\infty \to l_n^\infty$ and let $X_k := \mathrm{Im}(P)$. Since $x_s \in \mathbf{C}^k$ and $((x_s,x_t))$ contains a rank k submatrix for $s_2 = t_2 = k$, we have $\dim X_k = \mathrm{rank}(P) = k$. Let

$$T := P - (1/k - 1/k^{3/2})\mathrm{Id} : l_n^\infty \to l_n^\infty.$$

All entries of T have absolute value $\leqq k^{-3/2}$; thus we have

$$\nu(T) = \sum_t \sup_s |T_{st}| = k^2 k^{-3/2} = k^{1/2},$$

$$\begin{aligned}\mathrm{tr}(T: X_k \to X_k) &= \mathrm{tr}(P) - (1/k - 1/k^{3/2})\mathrm{tr}(\mathrm{Id}: X_k \to X_k)\\ &= k - 1 + 1/\sqrt{k}\end{aligned}$$

for the nuclear norm of T and the trace. Clearly

$$\gamma(X_k) \geqq \frac{\mathrm{tr}(T: X_k \to X_k)}{\nu(T)} = \sqrt{k} - 1/\sqrt{k} + 1/k.$$

To prove the reverse inequality, note that P is a projection because the vectors

$$y_j := k^{-1/2}\left(\exp\left(\frac{2\pi i}{k}\{s_1 j + s_2 j^2\}\right)\right)_{s_1,s_2=1}^{k} \in \mathbf{C}^n$$

satisfy $(y_j, y_l) = k\delta_{jl}$ since

$$\sum_{s_1=1}^{k} \exp\left(\frac{2\pi i}{k}(j-l)s_1\right) = 0, \qquad 1 \leqq j \neq l \leqq k.$$

Moreover, in each row of $P = (p_{st})$ there are $(k-1)$ zeros: fixing $s = (s_1, s_2)$, one has $p_{st} = 0$ for all $t = (t_1, s_2)$ with $1 \leqq t_1 \neq s_1 \leqq k$. Hence we find for $X_k \subseteq l_n^\infty$

$$\gamma(X_k) \leqq \|P\| = \max_s \sum_t |p_{st}| \leqq \frac{1}{k}(1 + (k^2 - k)/\sqrt{k}) = \sqrt{k} - 1/\sqrt{k} + 1/k.$$

If Y denotes the matrix with columns $y_1, \ldots, y_k$, we have rank $Y = k$ and

$$X_k = \mathrm{Im}(P) = \mathrm{Im}(YY^*) = \mathrm{Im}(Y),$$

i.e. X_k is spanned by the vectors $y_1, \ldots, y_k$. □

REMARK. The value of $|(x_i, x_j)| = \sqrt{(n-k)/k(n-1)} = 1/\sqrt{k+1}$ in (d) of Theorem 1 was not quite attained by our vectors x_s; instead, a few scalar products were 0, but most of them were of absolute value $1/\sqrt{k}$.

We now turn to the real case. The problem of equiangular lines and its connection to graph theory, group theory and combinatorial geometry has been studied extensively, cf. Lemmens–Seidel [2], Seidel [5], and Taylor [6].

THEOREM 3. *Let $p \neq 2$ be a prime, $q = p^m$ for some $m \in \mathbf{N}$, $k = q^2 - q + 1$, $n = q^3 + 1$. Then there is a real k-dimensional subspace X_k of l_n^∞ with projection constant*

$$\gamma(X_k) = f(k, n) = \frac{q^2+1}{q+1} \geqq \sqrt{k} - \frac{1}{2}(1 + 1/k).$$

PROOF. Taylor ([6], Theorem 6.1, Example 6.5) constructed n equiangular vectors $x_i \in \mathbf{R}^k$ satisfying (d) of Theorem 1,

$$(3) \qquad \|x_i\|_2 = 1, \quad |(x_i, x_j)| = 1/q = \sqrt{(n-k)/k(n-1)}, \quad i \neq j,$$

for the above values of k and n, cf. also [2] and [5] for the translation of the graph- and group-theoretical language into (3). Theorem 1 is directly applicable. □

There seems to be no explicit description of the vectors x_i and thus of the vectors spanning X_k in this case, although the existence is guaranteed. A somewhat more concrete construction was given by Lemmens–Seidel [2], Theorem 3.1, which for the projection constants yields similar information, although (d) of Theorem 1 is not quite satisfied:

THEOREM 3′. *Let* $q = 2^m$, $m \in \mathbf{N}$, $k = q^2 + q + 1$, $n = q(q^2 + q + 1)$. *Then there is a real k-dimensional subspace X_k of l_n^∞ the projection constant of which satisfies*

$$\sqrt{k} - 1/2(1 + 1/\sqrt{k}) \leqq \gamma(X_k) \leqq \sqrt{k} - 1/2(1 - 4/\sqrt{k}).$$

PROOF. We first indicate Lemmens–Seidel's construction of vectors $x_i \in \mathbf{R}^k$ with $\|x_i\|_2 = 1$ and $|(x_i, x_j)| = 1/(q+1)$ for $i \neq j$, $i, j = 1, \ldots, n$, cf. [2]. Let N be the line-point incidence matrix of the projective plane of order $q = 2^m$. This is a $k \times k$ matrix containing in each row and each column exactly $q + 1$ ones and zeros otherwise; mutual scalar products of different rows are one. Fix a $q \times q$ Hadamard-matrix W (with entries ± 1 and $W^2 = q$ Id, e.g. the Walsh matrices) and replace in each row of N exactly q of the ones by the q columns of W and the remaining $(q+1)$st one by a column of q ones. The zeros are replaced by q-columns of zeros. This yields a $qk \times k = n \times k$ matrix $\tilde{M}$. Let $M = 1/\sqrt{q+1}\tilde{M}$. The n rows $x_1, \ldots, x_n \in \mathbf{R}^k$ of M then satisfy $\|x_i\|_2 = 1$, $|(x_i, x_j)| = 1/(q+1)$ for $i \neq j$. M has rank k since already N has rank k because

$$(\det N)^2 = \det(NN^t) = \det\begin{pmatrix} q+1 & & 1 \\ & \ddots & \\ 1 & & q+1 \end{pmatrix} = q^{k-1}(q+1)^2 \neq 0.$$

Let $P = 1/qMM^t = 1/q((x_i, x_j))_{i,j=1}^n : l_n^\infty \to l_n^\infty$ and $X_k := \operatorname{Im}(P) \subseteq l_n^\infty$. Since rank $M = k$, X_k is k-dimensional and spanned by the columns of $\tilde{M}$, since $X_k = \operatorname{Im}(MM^t) = \operatorname{Im}(M)$. Each of these columns consists of $q^2 + q$ elements ± 1 and q^3 elements 0. In simple cases, e.g. $k = 7$, $n = 4$, it is easy to write down these spanning vectors explicitly. Anyhow, consider

$$T = P - (1/q - 1/q(q+1))\mathrm{Id} = (t_{ij}) : l_n^\infty \to l_n^\infty.$$

Then $|t_{ij}| = 1/q(q+1)$ for all $i, j = 1, \ldots, n$. Similarly, as in the proof of Theorem 2, we find

$$\gamma(X_k) \geqq \frac{\operatorname{tr}(T : X_k \to X_k)}{\nu(T)} \geqq \frac{n/q - (1/q - 1/q(q+1))k}{n/q(q+1)}$$

$$= q \geqq \sqrt{k} - 1/2(1 + 1/\sqrt{k}).$$

The upper bound is by the general formula (1)

$$\gamma(X_k) \leqq f(k,n) \leqq q + 1/q \leqq \sqrt{k} - 1/2(1 - 4/\sqrt{k}).$$

By the way, P is not a projection in this case although it is very close to one in the sense that it has eigenvalues $1 \pm 1/q$, each $(k-1)/2$-fold, one eigenvalue 1 and the remaining eigenvalues 0. □

In both real cases, $n \sim k^{3/2}$. Since by remark (ii) after Theorem 1 we have necessarily $n \leqq k(k+1)/2$ ($\mathbf{R}$) or $n \leqq k^2$ ($\mathbf{C}$) if (a) to (e) of Theorem 1 hold, for larger values of n it follows that

$$\gamma(X_k, Y_n) < f(k,n).$$

Since $f(k,n)$ is increasing in n, this motivates the

QUESTION. It is true that for any k-dimensional space X_k

$$\gamma(X_k) \leqq f(k, n(k)) \qquad \text{with } n(k) := \begin{Bmatrix} k(k+1)/2 & \mathbf{R} \\ k^2 & \mathbf{C} \end{Bmatrix}?$$

Even for $k=2$, $\gamma(X_2^{\mathbf{R}}) \leqq 4/3$ and $\gamma(X_2^{\mathbf{C}}) \leqq (1+\sqrt{3})/2$ seems to be unknown; there are spaces attaining these bounds. A positive answer would mean that $\gamma(X_k) \leqq \sqrt{k} - c/\sqrt{k}$ for some absolute constant $c > 0$ in the real and complex case, i.e. the example of Theorem 2 would be in a sense worst possible.

Note added in proof. The construction in Theorem 3′ can be modified to find spaces X_k with 1-unconditional basis with $\lambda(X_k) \geqq \sqrt{k} - 1$. In fact, if $(z_j)_{j=1,\dots,k}$ are the rows of the incidence matrix N, let X_k denote $\mathbf{R}^k$ equipped with the norm

$$\|x\| := \sup_{1 \leqq j \leqq k} \langle z_j, |x| \rangle.$$

Clearly X_k inbeds isometrically into $l_k^\infty(l_k^1)$, and the relative projection constant of X_k in this space is $\geqq q \geqq \sqrt{k} - 1$. This can be seen by replacing T in the proof of Theorem 3′ by

$$T = (z_{ji} z_{kl}(1 - \delta_{jk})\delta_{il})_{ji,kl} : l_k^\infty(l_k^1) \to l_k^\infty(l_k^1).$$

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